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Non-uniqueness of quantized Yang–Mills theories

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Abstract. We consider quantized Yang–Mills theories in the framework of causal perturbation theory which goes back to Epstein and Glaser. In this approach gauge invariance is expressed by a simple commutator relation for the S-matrix. The most general coupling which is gauge invariant to first order contains a two-parametric ambiguity in the ghost sector: a divergence- and a coboundary-coupling may be added. We prove (not completely) that the higher orders with these two additional couplings are also gauge invariant. Moreover, we show that the ambiguities of the n-point distributions restricted to the physical subspace are only a sum of the divergences (in the sense of vector analysis). It turns out that the theory without divergence- and coboundary-coupling is the simplest one in a quite technical sense. The proofs for the n-point distributions containing coboundary-couplings are given up to third or fourth order only, whereas the statements about the divergence-coupling are proved for all orders.

1. Introduction

1.1. The model

In a recent series of papers [1–5] non-Abelian gauge invariance has been studied in the framework of causal perturbation theory [6, 7]. This approach, which goes back to Epstein and Glaser [6], has the merit that one works exclusively with free fields, which are mathematically well-defined, and that one performs only justified operations with them.

In causal perturbation theory one makes an ansatz for the S-matrix as a formal power series in the coupling constant

$$S(g_0, g_1, \ldots, g_l) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{i_1,\ldots,i_n=0}^{l} \int \mathrm{d}^4x_1 \cdots \mathrm{d}^4x_n \; T_n^{i_1\ldots i_n}(x_1, \ldots, x_n) g_{i_1}(x_1) \cdots g_{i_n}(x_n). \tag{1.1}$$

The indices $i \in \{0, 1, \ldots, l\}$ label different couplings T_1^i, which are switched by different test functions $g_i \in \mathcal{S}(\mathbb{R}^4)$. The operator-valued distribution $T_n^{i_1\ldots i_n}(x_1, \ldots, x_n)$ has a vertex of the type $T_1^{i_s}$ at x_s $(1 \leqslant s \leqslant n)$. The T_n's are constructed inductively from the given first order (see appendix A). In our model the $i = 0$-coupling

$$T_1^0(x) \stackrel{\text{def}}{=} T_1^{0A}(x) + T_1^{0u}(x) \tag{1.2}$$

is the usual three-gluon coupling

$$T_1^{0A}(x) \stackrel{\text{def}}{=} \tfrac{1}{2} i g f_{abc} : A_{\mu a}(x) A_{\nu b}(x) F_c^{\nu\mu}(x) : \tag{1.3}$$

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plus the usual ghost coupling

$$T_1^{0u}(x) \stackrel{\text{def}}{=} -igf_{abc} : A_{\mu a}(x) u_b(x) \partial^\mu \tilde{u}_c(x) : . \tag{1.4}$$

Here g is the coupling constant and f_{abc} are the structure constants of the group SU(N). The gauge potentials A_a^μ, $F_a^{\mu\nu} \stackrel{\text{def}}{=} \partial^\mu A_a^\nu - \partial^\nu A_a^\mu$, and the ghost fields u_a, $\tilde{u}_a$ are massless and fulfil the wave equation. (We work throughout in the Feynman gauge $\lambda = 1$.)

Gauge invariance means roughly speaking that the commutator of the $T_n^{0\ldots0}$-distributions with the gauge charge

$$Q \stackrel{\text{def}}{=} \int_{t=\text{constant}} \mathrm{d}^3x \; (\partial_\nu A_a^\nu \overleftrightarrow{\partial}_0 u_a) \tag{1.5}$$

is a (sum of) divergence(s) (in the sense of vector analysis). To first order the following relation holds:

$$[Q, T_1^0(x)] = i\partial_\nu T_1^{1\nu}(x) \tag{1.6}$$

where

$$T_1^{1\nu}(x) \stackrel{\text{def}}{=} igf_{abc}[: A_{\mu a}(x) u_b(x) F_c^{\nu\mu}(x) : -\tfrac{1}{2} : u_a(x) u_b(x) \partial^\nu \tilde{u}_c(x) :]. \tag{1.7}$$

We choose this expression to be the $i = 1$-coupling in (1.1) and call it a Q-vertex. Note that only $[Q, T_1^{0A}]$ is not a divergence. In order to have gauge invariance to first order, we are forced to introduce the ghost coupling T_1^{0u}, equation (1.4). However, the latter coupling is not uniquely fixed by this procedure. The present paper deals with these ambiguities. We define gauge invariance in arbitrary order [2] by

$$[Q, T_n^{0\ldots0}(x_1, \ldots, x_n)] = i \sum_{l=1}^{n} \partial_\nu^{x_l} T_n^{0\ldots010\ldots0\nu}(x_1, \ldots, x_n) \tag{1.8}$$

where the upper index 1 in $T_n^{0\ldots010\ldots0}$ is at the lth position. The divergences on the right-hand side of (1.8) are precisely specified: $T_n^{0\ldots010\ldots0}(x_1, \ldots, x_n)$ is the T_n-distribution of (1.1) which has a Q-vertex (1.7) at x_l and all other vertices are T_1^0-couplings, equation (1.2). Gauge invariance (1.8), which has been proved for all orders n [1–5], implies the invariance of the S-matrix $S(g, 0, \ldots, 0)$ (1.1) with respect to simple gauge transformations of the *free* fields [5]. These transformations are the *free field version of the famous BRS transformations* [8]. Moreover, *unitarity on the physical subspace* [4] can be proved by means of gauge invariance (1.8). The C-number identities expressing (1.8) imply the *Slavnov–Taylor identities* [9]. Finally we mention that the four-gluon interaction is a second order normalization term, which is uniquely fixed by gauge invariance (see [1, 5] and equation (2.59)).

Let us turn to the above-mentioned non-uniqueness in the ghost sector. The most popular method for deriving the ghost coupling is that of Faddeev and Popov. However, this method of quantization contains loopholes (even in perturbation theory) [10]. Therefore, Beaulieu [10] determined the quantum Lagrangian from the requirement of its full BRS invariance. We proceed in an analogous way. *Our aim is to work out the most general Yang–Mills theory which is gauge invariant (1.8) for all orders and to investigate the physical and technical implications of the ambiguities.*

1.2. The most general coupling which is gauge invariant to first order

In order to simplify the notation we define an operator d_Q by means of our gauge charge Q (1.5)

$$d_Q A \stackrel{\text{def}}{=} QA - (-1)^{Q_g} A (-1)^{Q_g} Q \tag{1.9}$$

where Q_g is the ghost charge operator [11, 12]

$$Q_g \stackrel{\text{def}}{=} i \int_{t=\text{constant}} \mathrm{d}^3x \; : \tilde{u}_a(x) \overleftrightarrow{\partial}_0 u_a(x) : \qquad [Q_g, u_a] = -u_a \quad [Q_g, \tilde{u}_a] = \tilde{u}_a. \tag{1.10}$$

and A is a suitable operator on the Fock space such that equation (1.9) makes sense. If the ghost charge of A is an integer, $[Q_g, A] = zA$, $z \in \mathbb{Z}$, the expression (1.9) is the commutator or anticommutator of Q with A. Note the product rule

$$d_Q(AB) = (d_Q A)B + (-1)^{Q_g} A (-1)^{Q_g} d_Q B. \tag{1.11}$$

One easily verifies [1] that

$$Q^2 = 0 \tag{1.12}$$

which implies that

$$(d_Q)^2 = 0. \tag{1.13}$$

Because d_Q is nilpotent, it can be interpreted as coboundary-operator in the framework of a homological algebra [11]. (The gradiation is given by the ghost charge (1.10).) Therefore, we call an element of the range (kernel) of d_Q a coboundary (cocycle).

Let us add a coboundary

$$\beta_1 d_Q K_1(x) \qquad \beta_1 \in \mathbb{R} \;\; \text{arbitrary} \tag{1.14}$$

with

$$K_1(x) \stackrel{\text{def}}{=} g f_{abc} : u_a(x) \tilde{u}_b(x) \tilde{u}_c(x) : \tag{1.15}$$

to $T_1^0(x)$. Due to (1.13), gauge invariance to first order (1.6) remains true with the same Q-vertex $T_1^{1\nu}$ (1.7). Moreover, we add a divergence

$$\beta_2 \partial_\mu K_2^\mu(x) \qquad \beta_2 \in \mathbb{R} \;\; \text{arbitrary} \tag{1.16}$$

with

$$K_2^\mu(x) \stackrel{\text{def}}{=} i g f_{abc} : A_a^\mu(x) u_b(x) \tilde{u}_c(x) : \tag{1.17}$$

to $T_1^0(x)$. Simultaneously adding $\beta_2 d_Q K_2^\nu(x)$ to $T_1^{1\nu}(x)$, our gauge invariance (1.6) is obviously preserved. Are there further couplings which are gauge invariant to first order? The answer is 'no' [11, 13], if the following, physically reasonable requirements are additionally imposed.

(A) The coupling is a combination of at least three free field operators.
(B) The coupling has mass-dimension $\leqslant 4$. This guarantees (re)normalizability of the theory, if the fundamental (anti)commutators have singular order $\omega([A_a^\mu, A_b^\nu]) = -2$ and $\omega(\{u_a, \tilde{u}_b\}) = -2$ (see appendix A and [2]).
(C) Lorentz covariance.
(D) SU(N)-invariance.
(E) The coupling has ghost charge zero: $[Q_g, T_1^0] = 0$.
(F) Invariance with respect to the discrete symmetry transformations P, T and C.
(G) Pseudo-unitarity $S_1(g_0^*, 0, \ldots, 0)^K = S_1(g_0, 0, \ldots, 0)^{-1}$ forces β_1, β_2 to be real. (S_1 is the first order $n = 1$ of (1.1) and K is a conjugation which is related to the adjoint [4, 12].)

Remarks. (1) The self-interaction of the gauge bosons T_1^A (1.3) is unique. There is only an ambiguity in the ghost coupling.

(2) In [5] the coupling to fermionic matter fields in the fundamental representation was studied in detail. It is easy to see that the above requirements fix this coupling uniquely. Therefore, we do not consider matter fields in this paper.

1.3. Outline of the paper

The paper yields the following results.

(A) The higher orders with divergence- or coboundary-coupling (1.14)–(1.17) are gauge invariant for all values of $\beta_1, \beta_2 \in \mathbb{R}$ (sections 2.2 and 2.4). (For the coboundary-coupling this will be proved up to third order only.) The analogous result for the full BRS symmetry in the usual Lagrangian approach is known in the literature, see, e.g., [10]. However, only a one-parametric ambiguity is studied there. This difference will be discussed in remark (4) of section 2.7.

(B) We will show that the T_n's with divergence-coupling are divergences with respect to their divergence-vertices (section 2.2). The T_n's ($1 \leqslant n \leqslant 4$) with coboundary-coupling are divergences too, if they are restricted to the physical subspace [4] (section 2.8). This will be an immediate consequence of a representation of these T_n's, which will be proved in section 2.4.

(C) The results at higher orders about the divergence-coupling and partly the results about the coboundary-coupling are independent on the explicit expressions (1.2)–(1.4) and (1.14)–(1.17) of the couplings (section 2.5). They apply to any gauge-invariant quantum field theory.

(D) Gauge invariance for second-order tree diagrams requires normalization terms, namely the usual four-gluon interaction and a four-ghost interaction (section 2.7). (The latter appears only for $(\beta_1, \beta_2) \neq (0, 0)$.) By studying these normalization terms we will find a criterion which reduces the freedom in the choice of $\beta_1, \beta_2 \in \mathbb{R}$ to a one-parametric set (sections 2.7 and 2.8). We will mention a second, quite technical criterion which gives another restriction of β_1, β_2 (section 2.8). Together we will see that the theory with $\beta_1 = 0 = \beta_2$ is the simplest one.

(E) The Q-vertex is not uniquely fixed by gauge invariance to first order, equation (1.6). In order to prove gauge invariance at *higher* orders of the theory $(T_1^0 + \beta_1 d_Q K_1 + \beta_2 \partial_\mu K_2^\mu)$, $\beta_1, \beta_2 \in \mathbb{R}$ (equations (1.2)–(1.4), (1.14), (1.16)), it is not necessary to modify the above introduced Q-vertex (equation (1.7) plus $\beta_2 d_Q K_2^\nu$). Therefore, the ambiguity of the Q-vertex is not very interesting. Nevertheless, we show in section 2.3 that the possible modifications of the Q-vertex do not destroy gauge invariance at higher orders.

(F) In appendix C we assume that certain identities hold. They exclusively concern the starting-coupling T_1^0 (1.2)–(1.4), its Q-vertex T_1^1 (1.7) and its 'Q-Q-vertex' T_1^5 introduced below (2.5), and are a kind of generalization of gauge invariance (1.8). A special case of this assumption is verified in appendix B. By means of these identities we will be able to prove the results about the coboundary-coupling for *all* orders.

2. Divergence- and coboundary-coupling at higher orders

2.1. Preparations

In order to study the T_n's with a divergence- (1.16) and/or a coboundary-coupling (1.14) at higher orders $n \geqslant 2$, we define a big theory which contains these couplings and some auxiliary vertices

$$S_1(g_0, g_1, \ldots, g_7) \stackrel{\text{def}}{=} \int \mathrm{d}^4x\ \{T_1^0(x)g_0(x) + T_1^{1\nu}(x)g_{1\nu}(x) + T_1^2(x)g_2(x) + T_1^{3\nu}(x)g_\nu(x)$$

$$+ T_1^{4\nu}(x)g_{4\nu}(x) + T_1^{5\nu\mu}(x)g_{5\nu\mu}(x) + T_1^6(x)g_6(x) + T_1^7(x)g_7(x)\} \tag{2.1}$$

where T_1^0, $T_1^{1\nu}$ are given by (1.2)–(1.4) and (1.7); furthermore,

$$T_1^{4\nu}(x) \stackrel{\text{def}}{=} \beta_2 K_2^{\nu}(x) \tag{2.2}$$

$$T_1^2(x) \stackrel{\text{def}}{=} \partial_\nu T_1^{4\nu}(x) = \beta_2 \partial_\nu K_2^{\nu}(x) \tag{2.3}$$

$$iT_1^{3\nu}(x) \stackrel{\text{def}}{=} d_Q T_1^{4\nu}(x) = \beta_2 d_Q K_2^{\nu}(x) \tag{2.4}$$

$$T_1^{5\nu\mu}(x) \stackrel{\text{def}}{=} \tfrac{1}{2} igf_{abc} : u_a(x) u_b(x) F_c^{\nu\mu}(x) := -T_1^{5\mu\nu}(x) \tag{2.5}$$

$$T_1^6(x) \stackrel{\text{def}}{=} \beta_1 K_1(x) \tag{2.6}$$

and

$$T_1^7(x) \stackrel{\text{def}}{=} d_Q T_1^6(x) = \beta_1 d_Q K_1(x). \tag{2.7}$$

For technical reasons the divergence-coupling T_1^2 (2.3) and the coboundary-coupling T_1^7 (2.7) are not directly added to T_1^0; they are both smeared out with a separate test function. The appearance of the vertex $T_1^{5\nu\mu}$ is motivated by the relation

$$d_Q T_1^{1\nu}(x) = i\partial_\mu T_1^{5\nu\mu}(x). \tag{2.8}$$

Therefore, we sometimes call T_1^5 the 'Q-Q-vertex'. Furthermore, note that $T_1^{5\nu\mu}$ is a cocycle

$$d_Q T_1^{5\nu\mu}(x) = 0. \tag{2.9}$$

The vertices $T_1^{1\nu}$, $T_1^{3\nu}$ and T_1^6 are fermionic; all other vertices are bosonic. The first ones give rise to some additional minus signs in the inductive construction of the T_n's, but there is no serious complication (see the appendix of [3]). We are interested in the physically relevant theory

$$T_n(x_1, \ldots, x_n) \stackrel{\text{def}}{=} \sum_{i_1,\ldots,i_n \in \{0,2,7\}} T_n^{i_1\ldots i_n}(x_1, \ldots, x_n) \tag{2.10}$$

which corresponds to the choice $g \stackrel{\text{def}}{=} g_0 = g_2 = g_7 \neq 0$ and $g_1 = 0$, $g_{3\nu} = 0$, $g_{4\nu} = 0$, $g_{5\nu\mu} = 0$ and $g_6 = 0$ in the nth-order S-matrix $S_n(g_0, g_1, \ldots, g_7)$. Gauge invariance in the sense (1.8) of this theory is formulated in terms of the Q-vertices $T_1^{1\nu}$, $T_1^{3\nu}$ and $T_1^{8\nu} \stackrel{\text{def}}{=} 0$. This means that to first order

$$d_Q T_1^0 = i\partial_\nu T_1^{1\nu} \tag{2.11}$$

$$d_Q T_1^2 = i\partial_\nu T_1^{3\nu} \tag{2.12}$$

$$d_Q T_1^7 = 0 \tag{2.13}$$

and that to arbitrary order n

$$d_Q T_n^{i_1\ldots i_n} = i \sum_{l=1}^{n} \partial_\nu^l T_n^{i_1\ldots i_{l-1}\, i_l+1\, i_{l+1}\ldots i_n\, \nu} \tag{2.14}$$

where $i_1, \ldots, i_n \in \{0, 2, 7\}$ and

$$T_n^{i_1\ldots 8\ldots i_n\, \nu} \stackrel{\text{def}}{=} 0. \tag{2.15}$$

We shall often use the property that $T_n^{0\ldots 0}$ is gauge invariant (1.8) [1–5].

2.2. Higher orders with divergence-coupling

We are going to prove the following proposition.

Proposition 1. Choosing suitable normalizations, the relations

$$F_n^{2\ldots 20\ldots 0}(x_1,\ldots,x_n) = \partial^1_{\mu_1}\cdots\partial^r_{\mu_r} F_n^{4\ldots 40\ldots 0\mu_1\ldots\mu_r}(x_1,\ldots,x_n) \tag{2.16}$$

$$F_n^{32\ldots 20\ldots 0\nu}(x_1,\ldots,x_n) = \partial^2_{\mu_2}\cdots\partial^r_{\mu_r} F_n^{34\ldots 40\ldots 0\nu\mu_2\ldots\mu_r}(x_1,\ldots,x_n) \tag{2.17}$$

$$F_n^{2\ldots 210\ldots 0\nu}(x_1,\ldots,x_n) = \partial^1_{\mu_1}\cdots\partial^r_{\mu_r} F_n^{4\ldots 410\ldots 0\mu_1\ldots\mu_r\nu}(x_1,\ldots,x_n) \tag{2.18}$$

hold for all $F = A', R', R'', D, A, R, T', T, \tilde{T}$ and to all orders n.

Remarks. (1) The assertions (2.16)–(2.18) are generalizations of (2.3) to arbitrary orders and mean that the divergence-structure of T_1^2 can be maintained by constructing the higher orders.

(2) Due to the symmetrization (A.14) the $T_n^{\cdots}$, $\tilde{T}_n^{\cdots}$ fulfil

$$T_n^{i_1\ldots i_n}(x_1,\ldots,x_n) = (-1)^{f(\pi)} T_n^{i_{\pi 1}\ldots i_{\pi n}}(x_{\pi 1},\ldots,x_{\pi n}) \qquad \forall\ \pi\in\mathcal{S}_n \tag{2.19}$$

where the Lorentz indices are also permuted, and $f(\pi)$ is the number of transpositions of fermionic vertices in π. Therefore, equations (2.16)–(2.18) remain true for T_n, $\tilde{T}_n$, if the indices are permuted according to (2.19).

(3) We will see in the proof that the $T_n^{\ldots 4\ldots}$'s on the right-hand side can be normalized in an arbitrary symmetrical way. (A normalization is said to be symmetrical if the corresponding $T_n^{\cdots}$ satisfies (2.19).) However, the normalization of the $T_n^{\ldots 2\ldots}$'s on the left-hand side is uniquely fixed by the normalization of the $T_n^{\ldots 4\ldots}$'s.

Proof. We show that equations (2.16)–(2.18) can be maintained in the inductive step $(n-1)\to n$ described in appendix A. Obviously there are only two operations in this step which need an investigation, namely (A) the construction of the tensor products in A'_n, R'_n, R''_n (equations (A.1)–(A.3)) and (B) the distribution splitting $D_n = R_n - A_n$ (equations (A.7)).

(A) Let us consider equation (2.17) for $A_n'^{\cdots}$ (equation (A.2))

$$A_n'^{32\ldots 20\ldots 0\nu}(x_1,\ldots;x_n) = \sum_{X,Y,(x_1\in X)} \tilde{T}_k^{32\ldots 20\ldots 0\nu}(X)\, T_{n-k}^{2\ldots 20\ldots 0}(Y,x_n)$$
$$+ \sum_{X,Y,(x_1\in Y)} \tilde{T}_k^{2\ldots 20\ldots 0}(X)\, T_{n-k}^{32\ldots 20\ldots 0\nu}(Y,x_n). \tag{2.20}$$

Inserting the induction hypothesis (2.16), (2.17) for lower orders k, $n-k$, we obtain

$$(2.20) = \sum_{(x_1\in X)} \partial^2_{\mu_2}\cdots\partial^s_{\mu_s}\tilde{T}_k^{34\ldots 40\ldots 0\nu\mu_2\ldots\mu_s}(X)\,\partial^1_{\mu_{s+1}}\cdots\partial^{r-s}_{\mu_r} T_{n-k}^{4\ldots 40\ldots 0\mu_{s+1}\ldots\mu_r}(Y,x_n)$$
$$+ \sum_{(x_1\in Y)} \partial^1_{\mu_1}\cdots\partial^s_{\mu_s}\tilde{T}_k^{4\ldots 40\ldots 0\mu_1\ldots\mu_s}(X)\,\partial^2_{\mu_{s+2}}\cdots\partial^{r-s}_{\mu_r} T_{n-k}^{34\ldots 40\ldots 0\nu\mu_{s+2}\ldots\mu_r}(Y,x_n)$$
$$= \partial^2_{\mu_2}\cdots\partial^r_{\mu_r} A_n'^{34\ldots 40\ldots 0\nu\mu_2\ldots\mu_r}(x_1,\ldots,x_n). \tag{2.21}$$

The other verfications of (2.16)–(2.18) for A'_n, R'_n, R''_n are completely analogous.

(B) According to (A) the D_n's, equation (A.4), fulfil (2.16)–(2.18). Let $R_n^{34\ldots 40\ldots 0\nu\mu_2\ldots\mu_r}$ be an arbitrary splitting solution of $D_n^{34\ldots 40\ldots 0\nu\mu_2\ldots\mu_r}$. Then the definition

$$R_n^{32\ldots 20\ldots 0\nu}(x_1,\ldots,x_n) \stackrel{\text{def}}{=} \partial^2_{\mu_2}\cdots\partial^r_{\mu_r} R_n^{34\ldots 40\ldots 0\nu\mu_2\ldots\mu_r}(x_1,\ldots,x_n) \tag{2.22}$$

yields a splitting solution of $D_n^{32...20...0\nu}$, because $R_n^{32...20...0\nu}$ (equation (2.22)) has its support in $\Gamma^+_{n-1}(x_n)$ (equation (A.6)) and $R_n^{32...20...0\nu} = D_n^{32...20...0\nu}$ on $\Gamma^+_{n-1}(x_n) \setminus \{(x_n, \ldots, x_n)\}$. The procedure for equations (2.16), (2.18) is similar. □

Applying d_Q to (2.16) we see that $d_Q T_n^{2...20...0}$ is a divergence

$$d_Q T_n^{2...20...0}(x_1, \ldots, x_n) = \partial^1_{\mu_1} \ldots \partial^r_{\mu_r} d_Q T_n^{4...40...0\mu_1\ldots\mu_r}(x_1, \ldots, x_n) \tag{2.23}$$

if there is at least one divergence-vertex T_1^2. However, the divergences on the right-hand side of (2.23) are derivatives with respect to the divergence-vertices and generally not with respect to the Q-vertices. Consequently, equation (2.23) does not mean gauge invariance of $T_n^{2...20...0}$ in the sense of (1.8) ((2.14)). In order to obtain the latter we will prove the following proposition.

Proposition 2. Starting with arbitrary symmetrical normalizations of $T_n^{4...40...0}$ and $T_n^{4...410...0}, \ldots, T_n^{4...40...01}$, there exists a symmetrical normalization of $T_n^{34...40...0}, \ldots, T_n^{4...430...0}$ such that the equation

$$d_Q T_n^{4...40...0\mu_1\ldots\mu_r} = i[T_n^{34...40...0\mu_1\ldots\mu_r} + \cdots + T_n^{4...430...0\mu_1\ldots\mu_r}$$
$$+\partial_\nu^{r+1} T_n^{4...410...0\mu_1\ldots\mu_r\nu} + \cdots + \partial^n_\nu T_n^{4...40...01\mu_1\ldots\mu_r\nu}] \tag{2.24}$$

holds for all orders n and for $r = 1, 2, \ldots, n$ vertices T_1^4 and T_1^3, respectively.

Remarks. (1) The assertion (2.24) is a kind of gauge invariance equation, which is a generalization of (2.4) and (2.11) to higher orders.

(2) We will prove (2.24) for all F_n, $F = A', R', R'', D, A, R, T', T, \tilde{T}$ by induction on n.

(3) Applying $\partial^1_{\mu_1} \cdots \partial^r_{\mu_r}$ to (2.24) we obtain the following corollary by means of proposition 1.

Corollary 3. With the normalization of (2.16) the distributions $F_n^{2...20...0}$, $F = A', R', R'', D, A, R, T', T, \tilde{T}$ are gauge invariant, i.e. they fulfil (2.14).

Proof of proposition 2. The proof follows the inductive construction of the T_n's. Since (2.24) is a linear equation, we merely have to consider the same operations (A) (construction of tensor products) and (B) (distribution splitting) as in the proof of proposition 1.

(A) Inserting the induction hypothesis (2.24) or gauge invariance (1.8) into $d_Q \tilde{T}_k^{4...40...0}$ and $d_Q T_{n-k}^{4...40...0}$ in

$$d_Q A_n'^{4...40...0\mu_1\ldots\mu_r}(x_1, \ldots; x_n) = \sum_{X,Y} \Big[(d_Q \tilde{T}_k^{4...40...0\mu_1\ldots\mu_s}(X)) T_{n-k}^{4...40...0\mu_{s+1}\ldots\mu_r}(Y, x_n)$$
$$+ \tilde{T}_k^{4...40...0\mu_1\ldots\mu_s}(X) d_Q T_{n-k}^{4...40...0\mu_{s+1}\ldots\mu_r}(Y, x_n)\Big] \tag{2.25}$$

one easily obtains the result that the A_n'-distributions fulfil (2.24), and similarly this holds for R_n', R_n''.

(B) Let $R_n^{4...40...0}, R_n^{4...410...0}, \ldots, R_n^{4...40...01}, R_n^{434...40...0}, \ldots, R_n^{4...430...0}$ be arbitrary splitting solutions of $D_n^{4...40...0}$, $D_n^{4...410...0}, \ldots, D_n^{4...40...01}$, $D_n^{434...40...0}, \ldots, D_n^{4...430...0}$. Due to (A) the D_n-distributions fulfil (2.24). Since the operators d_Q and ∂^s_ν do not enlarge the support of the distribution to which they are applied, by the definition

$$i R_n^{34...40...0\mu_1\ldots\mu_r} \stackrel{\text{def}}{=} d_Q R_n^{4...40...0\mu_1\ldots\mu_r} - i\left[R_n^{434...40...0\mu_1\ldots\mu_r} + \cdots + R_n^{4...430...0\mu_1\ldots\mu_r}\right.$$
$$\left.+ \partial^{r+1}_\nu R_n^{4...410...0\mu_1\ldots\mu_r\nu} + \cdots + \partial^n_\nu R_n^{4...40...01\mu_1\ldots\mu_r\nu}\right] \tag{2.26}$$

we obtain a splitting solution of $i D_n^{34...40...0\mu_1\ldots\mu_r}$. Obviously the $T_n' \stackrel{\text{def}}{=} R_n - R_n'$-distributions fulfil (2.24) and this equation is maintained in the symmetrization $T_n' \to T_n$ (A.14). □

2.3. Non-uniqueness of the Q-vertex $T_1^{1\nu}$ at higher orders

The total Q-vertex $T^\nu_{1/1} \stackrel{\mathrm{def}}{=} T_1^{1\nu} + T_1^{3\nu}$ of the theory (2.10) is not uniquely fixed by gauge invariance in first-order $d_Q(T_1^0 + T_1^2 + T_1^7) = i\partial_\nu T^\nu_{1/1}$. One has the freedom to replace $T^\nu_{1/1}$ by

$$T^\nu_{1/1\,B} \stackrel{\mathrm{def}}{=} T^\nu_{1/1} + \gamma B^\nu \qquad \gamma \in \mathbb{C} \quad \text{arbitrary} \tag{2.27}$$

if $\partial_\nu B^\nu = 0$. Requiring additionally that B^ν should fulfil the properties (A), (B), (C) and (D) listed in section 1.2, and have ghost charge -1, there remains only one possibility, namely

$$B^\nu(x) = \partial_\mu D^{\nu\mu}(x) \tag{2.28}$$

with

$$D^{\nu\mu}(x) \stackrel{\mathrm{def}}{=} igf_{abc} : u_a(x) A^\nu_b(x) A^\mu_c(x) := -D^{\mu\nu}(x). \tag{2.29}$$

This is proved in [11, 13]. The T_n-distribution with a modified Q-vertex $T^\nu_{1/1\,B}$ at x_l and with all other vertices being a $T_1 \stackrel{\mathrm{def}}{=} (T_1^0 + T_1^2 + T_1^7)$-coupling is denoted by $T^\nu_{n/l\,B}(x_1, \ldots, x_l, \ldots, x_n)$. That for an original Q-vertex $T^\nu_{1/1}$ is similarly denoted by $T^\nu_{n/l}(x_1, \ldots, x_n)$, that for a vertex B^ν by $B^\nu_{n/l}(x_1, \ldots, x_n)$, and that for $D^{\nu\mu}$ by $D^{\nu\mu}_{n/l}(x_1, \ldots, x_n)$. The relation $D^{\nu\mu} = -D^{\mu\nu}$ can be maintained in the inductive construction of the T_n's:

$$D^{\nu\mu}_{n/l} = -D^{\mu\nu}_{n/l}. \tag{2.30}$$

This is evident for the tensor products (A.1)–(A.3) and for the steps (A.4), (A.13)–(A.15). Concerning the splitting (A.7), note that the antisymmetrization (in $\nu \leftrightarrow \mu$) of an arbitrary splitting solution yields again a splitting solution. Due to proposition 1 (equation (2.16)), there exists a symmetrical normalization of $B^\nu_{n/l}$ which fulfils

$$B^\nu_{n/l} = \partial^l_\mu D^{\nu\mu}_{n/l}. \tag{2.31}$$

Moreover, the normalizations can be chosen such that (2.27) propagates to higher orders:

$$T^\nu_{n/l\,B} = T^\nu_{n/l} + \gamma B^\nu_{n/l}. \tag{2.32}$$

We conclude that

$$\partial^l_\nu T^\nu_{n/l\,B} = \partial^l_\nu T^\nu_{n/l}. \tag{2.33}$$

Assuming T_n, $T^\nu_{n/l}$ $(l = 1, \ldots, n)$ to be gauge invariant (i.e. to fulfil (1.8)), there exists a symmetrical normalization of the distributions $T^\nu_{n/l\,B}$, such that T_n, $T^\nu_{n/l\,B}$ are also gauge invariant. The modification (2.27) of the Q-vertex does not destroy gauge invariance at higher orders.

2.4. Higher orders with coboundary-coupling

The results of this section are summarized in the following proposition.

Proposition 4. Choosing suitable symmetrical normalizations the following statements hold for all $F = A', R', R'', D, T, \tilde{T}$:

At orders $1 \leqslant n \leqslant 4$ the F_n's with coboundary-coupling have the representation

$$F_n^{7\ldots70\ldots0} = \frac{1}{r}\{d_Q F_n^{67\ldots70\ldots0} + d_Q F_n^{767\ldots70\ldots0} + \cdots + d_Q F_n^{7\ldots760\ldots0}\} \tag{2.34a}$$

$$+\frac{i}{r}\sum_{l=r+1}^{n} \partial_\nu^l \{F_n^{67\ldots70\ldots010\ldots0\nu} + F_n^{767\ldots70\ldots010\ldots0\nu} + \cdots + F_n^{7\ldots760\ldots010\ldots0\nu}\} \tag{2.34b}$$

and they are gauge invariant (2.14) to orders $1 \leqslant n \leqslant 3$

$$d_Q F_n^{7\ldots70\ldots0} = i \sum_{l=r+1}^{n} \partial_\nu^l F_n^{7\ldots70\ldots010\ldots0\nu} \tag{2.35}$$

where each $F_n^{\cdots}$ has r upper indices 7 or 6, $1 \leqslant r \leqslant n$, and the upper index 1 is always at the lth position.

Equations (2.34), (2.35), the gauge invariance (1.8) of $T_n^{0\ldots0}$ ($n \in \mathbb{N}$) and the second-order identities

$$d_Q F_2^{16\nu} = i\partial_\mu^1 F_2^{56\nu\mu} - F_2^{17\nu} \tag{2.36}$$

$$d_Q F_2^{56\nu\mu} = F_2^{57\nu\mu} \tag{2.37}$$

$$d_Q F_2^{17\nu} = i\partial_\mu^1 F_2^{57\nu\mu} \tag{2.38}$$

$$d_Q F_2^{10\nu} = i\partial_\mu^1 F_2^{50\nu\mu} - i\partial_\mu^2 F_2^{11\nu\mu} \tag{2.39}$$

can all be fulfilled simultaneously.

Remarks. (1) Replacing $F_n^{i_1\ldots i_n}(x_1,\ldots,x_n)$ by $T_1^{i_1}(x_1)\cdots T_1^{i_n}(x_n)$ and applying (1.11), (1.13), (2.7)–(2.9), (2.11) and (2.13), equations (2.34)–(2.39) are obviously fulfilled—this is the intuition.

(2) Due to (2.19), similar equations with permuted upper indices hold for T_n, $\tilde{T}_n$.

(3) Applying d_Q to (2.34) we obtain

$$d_Q F_n^{7\ldots70\ldots0} = \frac{i}{r}\sum_{l=r+1}^{n} \partial_\nu^l \{d_Q F_n^{67\ldots70\ldots010\ldots0\nu} + \cdots + d_Q F_n^{7\ldots760\ldots010\ldots0\nu}\}. \tag{2.40}$$

However, this is not gauge invariance in the sense of the Q-vertices (2.14). The latter is given by (2.35).

(4) By means of (2.34), (2.35) the list (2.36)–(2.39) of second-order identities, which are a kind of gauge invariance equations, can be extended:

$$d_Q F_2^{70} = i\partial_\nu^2 F_2^{71\nu} \tag{2.41}$$

$$d_Q F_2^{77} = 0 \tag{2.42}$$

$$d_Q F_2^{60} = F_2^{70} - i\partial_\nu^2 F_2^{61\nu} \tag{2.43}$$

$$\tfrac{1}{2}(d_Q F_2^{67} + d_Q F_2^{76}) = F_2^{77}. \tag{2.44}$$

Proof of proposition 4. (A) Outline. The proof of (2.34), (2.35) is by induction for order n. However, we will see that the proof of (2.35) for order n needs identities of the type (2.36), (2.38), (2.39) at lower orders $k \leqslant n-1$. However, equation (2.39) cannot be proved by means of the general, elementary inductive methods of this section; it needs an explicit proof which uses the actual couplings (1.2)–(1.4), (1.7) and (2.5). This proof, which is given in appendix B, is similar to the proof of gauge invariance (1.8) of T_2^{00}. To prove an identity analogous to (2.39) at higher orders (see equation (2.50*a*) below), requires a huge amount of work (cf [2–5]), which is not done in this paper. Therefore, the inductive proof of gauge invariance (2.35) stops at $n = 3$. Moreover, the proof of (2.34) at order n needs (2.35) at lower orders $k \leqslant n-1$. Consequently, the representation (2.34) of $F_n^{7\ldots 70\ldots 0}$ will be proved for $n \leqslant 4$ only.

(B) Proof of (2.34) by means of (2.34), (2.35) at lower orders. We start with equation (A.2):

$$A_n'^{7\ldots 70\ldots 0}(x_1,\ldots;x_n) = \sum_{X,Y} \left\{ \frac{s}{r} \tilde{T}_k^{7\ldots 70\ldots 0}(X) T_{n-k}^{7\ldots 70\ldots 0}(Y,x_n) \right. \tag{2.45a}$$

$$\left. + \frac{r-s}{r} \tilde{T}_k^{7\ldots 70\ldots 0}(X) T_{n-k}^{7\ldots 70\ldots 0}(Y,x_n) \right\} \tag{2.45b}$$

where $\tilde{T}_k^{7\ldots 70\ldots 0}$ ($T_{n-k}^{7\ldots 70\ldots 0}$) has s ($r-s$) upper indices 7. Next we insert the induction hypothesis (2.34) for $\tilde{T}_k^{7\ldots 70\ldots 0}$ into (2.45*a*) (equation (2.34) for $T_{n-k}^{7\ldots 70\ldots 0}$ into (2.45*b*)). Then we apply (1.11) to the terms with a d_Q-operator and obtain

$$\frac{s}{r} \tilde{T}_k^{7\ldots 70\ldots 0}(X) T_{n-k}^{7\ldots 70\ldots 0}(Y,x_n) = \frac{1}{r} \left[d_Q(\tilde{T}_k^{67\ldots 70\ldots 0}(X) T_{n-k}^{7\ldots 70\ldots 0}(Y,x_n)) \right. \tag{2.46a}$$

$$+ \tilde{T}_k^{67\ldots 70\ldots 0}(X) d_Q T_{n-k}^{7\ldots 70\ldots 0}(Y,x_n) + \cdots + \tag{2.46b}$$

$$\left. + i \sum_{l=s+1}^{k} \{(\partial_\nu^l \tilde{T}_k^{67\ldots 70\ldots 010\ldots 0\nu}(X)) T_{n-k}^{7\ldots 70\ldots 0}(Y,x_n) + \cdots\} \right] \tag{2.46c}$$

and similarly for (2.45*b*). The next step is to insert the induction hypothesis (2.35) or gauge invariance (1.8) (the latter in the special case $r-s=0$) into $d_Q T_{n-k}^{7\ldots 70\ldots 0}(Y,x_n)$ in (2.46*b*). Then we see that the A_n'-distributions fulfil (2.34): the terms of type (2.46*a*) add up to (2.34*a*); (2.46*b*) and (2.46*c*) can be combined and all terms of this type give together (2.34*b*). Similarly one proves that the R_n'-, R_n''- and, therefore, the D_n-distributions satisfy (2.34).

We turn to the splitting (A.7). Let $R_n^{67\ldots 70\ldots 0}$, $R_n^{767\ldots 70\ldots 0}, \ldots, R_n^{67\ldots 70\ldots 010\ldots 0\nu}$, $R_n^{767\ldots 70\ldots 010\ldots 0\nu}, \ldots$ be arbitrary splitting solutions of the corresponding $D_n^{\cdots}$-distributions. By means of the definition

$$R_n^{7\ldots 70\ldots 0} \stackrel{\text{def}}{=} \frac{1}{r} \{d_Q R_n^{67\ldots 70\ldots 0} + d_Q R_n^{767\ldots 70\ldots 0} + \cdots\}$$

$$+ \frac{i}{r} \sum_{l=r+1}^{n} \partial_\nu^l \{R_n^{67\ldots 70\ldots 010\ldots 0\nu} + R_n^{767\ldots 70\ldots 010\ldots 0\nu} + \cdots\} \tag{2.34$'$}$$

we obtain a splitting solution of $D_n^{7\ldots 70\ldots 0}$, analogously to (2.22), (2.26). Obviously equation (2.34) is maintained in the remaining steps, namely the construction of T_n', T_n and $\tilde{T}_n$ (equations (A.13)–(A.15)).

(C) Proof of (2.35) by means of (2.34) for the same order n, and by means of (2.34), (2.35) and identities of the type (2.36), (2.38), (2.39) for lower orders. One can easily verify (by inserting (2.35) and (1.8) for lower orders) that the A_n'-, R_n'-, and R_n''-distributions fulfil (2.35). Therefore, as usual gauge invariance (2.35) can be violated in the distribution splitting only. However, to prove that this violation can be avoided by choosing a suitable normalization, is a completely non-trivial business [1–5]. Moreover, the normalization of

$T_n^{7...70...0}$ is restricted by (2.34′). Therefore, we use another route to prove (2.35) for T_n, $\tilde{T}_n$. We show that the right-hand side of (2.40) agrees with the right-hand side of (2.35), if a suitable symmetrical normalization of $T_n^{7...70...010...0\nu}$, $1 \leqslant r \leqslant n-1$, is chosen. (The case $r = n$ is trivial.) For this purpose we consider

$$A_n'^{7...70...010...0\nu} - \frac{1}{r}\{d_Q A_n'^{67...70...010...0\nu} + \cdots + d_Q A_n'^{7...760...010...0\nu}\} \tag{2.47}$$

where the upper index 1 is always at the lth position. We insert the definition (A.2) of the A_n'-distributions. Similarly to (2.25) we then apply (1.11) and the induction hypothesis, i.e. we insert (2.7), (2.8), (2.11) and (2.13) if $n = 2$, and additionally (2.36), (2.38), (2.39), (2.41)–(2.44) if $n = 3$. In this way we obtain

$$(2.47) = \frac{i}{r}\Bigg\{ \sum_{j=r+1\ (j\neq l)}^{n} [\pm\partial_\mu^j A_n'^{67...70...010...010...0\mu\nu} \pm \cdots \pm \partial_\mu^j A_n'^{7...760...010...010...0\mu\nu}] \tag{2.48a}$$

$$+\partial_\mu^l A_n'^{67...70...050...0\nu\mu} + \cdots + \partial_\mu^l A_n'^{7...760...050...0\nu\mu}\Bigg\}. \tag{2.48b}$$

In (2.48*a*) the two upper indices 1 are at the jth and lth positions, and we have a plus (minus) if $j < l$ $(j > l)$. One proves (2.47) = (2.48) for the R_n'-, R_n''-distributions in a similar way.

Analogously to (2.30), the antisymmetry $T_1^{5\nu\mu} = -T_1^{5\mu\nu}$, equation (2.5), can be preserved in the inductive construction of the T_n's. Starting with arbitrary splitting solutions $R_n^{67...70...050...0\nu\mu} = -R_n^{67...70...050...0\mu\nu}, \ldots, R_n^{7...760...050...0\nu\mu} = -R_n^{7...760...050...0\mu\nu}$, $R_n^{67...70...010...010...0}, \ldots, R_n^{7...760...010...010...0}$ we may (in a manner similar to that for (2.34′)) define $R_n^{7...70...010...0}$ by the equation (2.47)=(2.48) (with $A_n'^{...}$ everywhere replaced by $R_n^{...}$). This equation is not destroyed in the construction of T_n', T_n and $\tilde{T}_n$. Summing up, we have proved

$$\begin{aligned} F_n^{7...70...010...0\nu} &- \frac{1}{r}\{d_Q F_n^{67...70...010...0\nu} + \cdots + d_Q F_n^{7...760...010...0\nu}\} \\ &= \frac{i}{r}\Bigg\{ \sum_{j=r+1\ (j\neq l)}^{n} [\pm\partial_\mu^j F_n^{67...70...010...010...0\mu\nu} \pm \cdots \pm \partial_\mu^j F_n^{7...760...010...010...0\mu\nu}] \\ &\quad + \partial_\mu^l F_n^{67...70...050...0\nu\mu} + \cdots + \partial_\mu^l F_n^{7...760...050...0\nu\mu}\Bigg\} \end{aligned} \tag{2.49}$$

for all $F = A', R', R'', D, A, R, T', T, \tilde{T}$ and for $n \leqslant 3$, $1 \leqslant r \leqslant n-1$. We insert this equation into

$$\sum_{l=r+1}^{n} \partial_\nu^l \Bigg\{ F_n^{7...70...010...0\nu} - \frac{1}{r}[d_Q F_n^{67...70...010...0\nu} + \cdots + d_Q F_n^{7...760...010...0\nu}]\Bigg\} \tag{2.50}$$

for $F = T, \tilde{T}$. Taking the different signs of the (j,l)- and the (l,j)-term in $\sum_{j,l\ (j\neq l)} \pm\partial^l\partial^j F_n^{...010...010...}$ and $F_n^{...5...\nu\mu} = -F_n^{...5...\mu\nu}$ into account, we see that (2.50) vanishes. This is the desired result.

Proof of equations (2.36)–(2.39). The first identity (2.36) is the case $n = 2$, $r = 1$ of (2.49). All of equations (2.36)–(2.39) are easily verified for the $A_2'^{...}$-distributions, etc, and, therefore, can be violated only in the splitting. The latter is no problem for (2.37), since we may define $R_2^{57\nu\mu} \stackrel{\text{def}}{=} d_Q R_2^{56\nu\mu}$ for an arbitrary splitting solution R_2^{56}. Applying d_Q to (2.36), we obtain (2.38) by means of (2.37). Equation (2.39) remains, which is proved in appendix B by explicitly inserting the actual couplings. It turns out that there exists a normalization of $T_2^{10\nu}(x_1, x_2) = T_2^{01\nu}(x_2, x_1)$ such that (2.39) and gauge invariance (1.8) (to second order) are satisfied simultaneously. One easily verifies that this is the only problem of compatibility in (2.34)–(2.39) and (1.8). For example, to second order the distributions $T_2^{56\nu\mu} = -T_2^{56\mu\nu}$, T_2^{61}, T_2^{60}, T_2^{67} can be normalized in an arbitrary symmetrical way. Then the normalizations of T_2^{17}, T_2^{57}, T_2^{70}, T_2^{77} are uniquely fixed by (2.36), (2.37), (2.43), (2.44), and all identities (2.36)–(2.38) and (2.41)–(2.44) are fulfilled. The remaining distributions T_2^{00}, T_2^{10}, T_2^{50} and T_2^{11} appear only in (1.8) and (2.39). □

If the identities ($F = T, \ \tilde{T}$)

$$d_Q F_n^{5\ldots51\ldots10\ldots0} = i \sum_{j=t+1}^{t+s} (-1)^{(j-t-1)} \partial^j F_n^{5\ldots51\ldots151\ldots10\ldots0} + i(-1)^s \sum_{j=t+s+1}^{n} \partial^j F_n^{5\ldots51\ldots10\ldots010\ldots0}$$

$$n \in \mathbb{N} \quad 0 \leqslant t, s \leqslant n \quad t+s \leqslant n \tag{2.50a}$$

hold (where $F_n^{5\ldots51\ldots10\ldots0}$ on the left-hand side has t indices 5, s indices 1 and all derivatives on the right-hand side are divergences, the Lorentz indices being omitted), one can prove the representation (2.34) and gauge invariance (2.35) for all orders. This is shown in appendix C by a generalization of the proof shown here. Unfortunately, an inductive proof of (2.50*a*) by means of the simple technique of this section fails because of the splitting (A.7): there is no term in (2.50*a*) which has neither a d_Q-operator nor a derivative. We emphasize that the identities (2.50*a*) do not depend on the explicit form (1.14), (1.15) of the coboundary coupling (no upper indices 6 or 7 appear in (2.50*a*)). These identities concern solely the starting-coupling T_1^0, its Q-vertex T_1^1 and its Q-Q-vertex T_1^5.

Remark. The compatibility of (2.39) and gauge invariance (1.8) to second order is remarkable in the tree sector: each of these two identities fixes the normalization of $T_2^{10}|_{\text{tree}}$ *uniquely* and in fact these two normalizations agree (see appendix B and section 3.2 of [5]). This is a further hint that our gauge invariance (1.8) relies on a deeper (cohomological?) structure. Knowledge of the latter would presumably shorten the proof of (1.8) and would be an excellent tool for proving the missing identities (2.50*a*).

2.5. Generality of the results

In the preceeding sections 2.2 and 2.4, the explicit structures of the starting theory T_1^0 (1.2), of the corresponding Q-vertex $T_1^{1\nu}$ (1.7), of the divergence-coupling (1.16), (1.17) and of the coboundary-coupling (1.14), (1.15) have not been needed. We have used only the following properties.

(i) The starting theory T_1^0 is gauge invariant with respect to the Q-vertex $T_1^{1\nu}$ in all orders which are considered.
(ii) There exists a Q-Q-vertex $T_1^{5\nu\mu}(x)$ which fulfils

$$T_1^{5\nu\mu} = -T_1^{5\mu\nu} \qquad d_Q T_1^{5\nu\mu} = 0 \qquad d_Q T_1^{1\nu}(x) = i\partial_\mu T_1^{5\nu\mu}(x). \tag{2.51}$$

(iii) The second-order identity (2.39) holds and is compatible with gauge invariance (1.8) of T_2^{00}.

Only (i) is needed in section 2.2. Therefore, the results about the divergence-coupling apply to any gauge-invariant quantum field theory, e.g. to quantum gravity [14]. This also holds for (2.34) to second order, i.e. (2.43), (2.44).

If in addition (ii) is fulfilled ($d_Q T_1^5 = 0$ is not needed for the following statement), gauge invariance (2.35) is proved to second order (i.e. equations (2.41), (2.42) are valid), and this implies the identities (2.34) up to third order. Note that the modified Q-vertex $T^{\nu}_{1/1\,B}$ (2.27) also satisfies (ii):

$$d_Q T^{\nu}_{1/1\,B} = i\partial_\mu T^{5\nu\mu}_{1\,B} \tag{2.52}$$

with

$$T^{5\nu\mu}_{1\,B} \stackrel{\text{def}}{=} T^{5\nu\mu}_1 - i\gamma d_Q D^{\nu\mu} = -T^{5\mu\nu}_{1\,B} \qquad d_Q T^{5\nu\mu}_{1\,B} = 0. \tag{2.53}$$

For a model which satisfies (i), (ii) and all identities (2.50*a*) (equation (2.39) is a special case of the latter), the statements (2.34), (2.35) about the coboundary-coupling are also proved for all orders.

2.6. *n-point distributions with divergence- and coboundary-coupling*

The general case (2.10) of T_n containing the ordinary Yang–Mills coupling T^0_1, divergence-*and* coboundary-coupling can easily be traced back to the results of the preceeding sections 2.2, 2.4 and 2.5. We replace T^0_1 by

$$\bar{T}^0_1 \stackrel{\text{def}}{=} T^0_1 + T^2_1 = T^0_1 + \beta_2 \partial_\nu K^\nu_2 \tag{2.54}$$

and $T^{1\nu}_1$ by

$$\bar{T}^{1\nu}_1 \stackrel{\text{def}}{=} T^{1\nu}_1 + T^{3\nu}_1 = T^{1\nu}_1 - i\beta_2 d_Q K^\nu_2. \tag{2.55}$$

Due to corollary 3, the $\bar{T}^0_1$-theory is gauge invariant with respect to the Q-vertex $\bar{T}^{1\nu}_1$ in all orders, i.e. property (i) of section 2.5 is fulfilled. Obviously property (ii) also holds with the old T^5_1-vertex (2.5): $d_Q \bar{T}^{1\nu}_1 = i\partial_\mu T^{5\nu\mu}_1$. It would be very suprising if (2.39) would be wrong for the $(\bar{T}^0_1,\ \bar{T}^{1\nu}_1,\ T^{5\nu\mu}_1)$-couplings. By means of proposition 4 we conclude that the general n-point distributions (2.10) (with coboundary- *and* divergence-coupling) are gauge invariant to second (and most probably to third order), and we obtain the representation (2.34) with respect to the coboundary-vertices up to third (fourth) order.

Let us describe an alternative method. We replace T^0_1 by

$$\bar{T}^0_1 \stackrel{\text{def}}{=} T^0_1 + T^7_1 = T^0_1 + \beta_1 d_Q K_1. \tag{2.56}$$

The Q-vertex (1.7) needs no change: $d_Q \bar{T}^0_1 = i\partial_\nu T^{1\nu}_1$. Proposition 4 (2.35) tells us that the $\bar{T}^0_1$-theory is gauge invariant up to third order. Applying corollary 3 we obtain gauge invariance (2.14) of the general T_n's (2.10) up to third order. Moreover, due to proposition 1, these distributions are divergences with respect to their divergence-vertices at any order.

2.7. *Gauge-invariant normalization of second-order tree diagrams*

We only consider the tree sector and start with the following normalization of $T_2(x_1, x_2)$ (2.10) ($T_2 \stackrel{\text{def}}{=} T^{00}_2 + T^{20}_2 + T^{02}_2 + T^{22}_2 + T^{70}_2 + T^{07}_2 + T^{77}_2 + T^{27}_2 + T^{72}_2$). The C-number distributions of $T_{20}|_{\text{tree}}$ (the lower index 0 indicates this special normalization) are

$$t_{\mathcal{O}}(x_1 - x_2) \sim D^F(x_1 - x_2),\ \ \partial^\mu D^F(x_1 - x_2),\ \ \partial^\mu \partial^\nu D^F(x_1 - x_2) \tag{2.57}$$

and they have no local terms. The singular order ω of $t_{\mathcal{O}}$ (the number of derivatives on D^F in (2.57)) can be computed from the combination $\mathcal{O}$ of the four external free field operators (see $\omega(\mathcal{O})$ in (A.17)) and is $\omega(\mathcal{O}) = -2, -1, 0$. For each four-leg combination $\mathcal{O}$ with $\omega(\mathcal{O}) = 0$ we may add a local term

$$N_{\mathcal{O}}(x_1 - x_2) = C_{\mathcal{O}} \delta(x_1 - x_2) : \mathcal{O}(x_1 - x_2) : \tag{2.58}$$

to T_{20}, where $C_{\mathcal{O}}$ is a free normalization constant (A.12). Gauge invariance (2.14) fixes the values of $C_{\mathcal{O}}$ uniquely [1, 5, 13]. In T_2^{00} the normalization term

$$N_{AAAA}(x_1 - x_2) = -\tfrac{1}{2} i g^2 f_{abr} f_{cdr} \delta(x_1 - x_2) : A_{\mu a} A_{\nu b} A_c^\mu A_d^\nu : \tag{2.59}$$

is required [1, 5]. This is the four-gluon interaction, which propagates to higher orders in the inductive construction of the T_n's (see section 4.2 of [15]). The normalization terms (2.58) of $T_2^{20}, \ldots, T_2^{72}$ which are needed for gauge invariance (2.14) can quickly be calculated by using our results. We have proved that T_2^{20}, T_2^{22}, T_2^{70}, T_2^{77} and T_2^{27} are gauge invariant with the normalizations given by proposition 1, equation (2.16) (proposition 4, equation (2.34)). (In the case of T_2^{27} we perform the replacement (2.54), (2.55) (or alternatively (2.56)), before applying (2.34) (equation (2.16))). Therefore, we simply have to pick out the local terms in $\partial_\mu^1 T_2^{40\mu} (= T_2^{20})$, $\partial_\mu^1 \partial_\nu^2 T_2^{44\mu\nu} (= T_2^{22})$, $d_Q T_2^{60} + i\partial_\nu^2 T_2^{61\nu} (= T_2^{70})$ and in $\frac{1}{2}(d_Q T_2^{67} + d_Q T_2^{76}) (= T_2^{77})$. In the tree sector there are no local terms in T_2^{40}, T_2^{44}, T_2^{60}, T_2^{61}, T_2^{67} (their normalization is unique) and, therefore, neither are there any in $d_Q T_2^{60}$, $d_Q T_2^{67}$. All local terms are generated by the divergences in $\partial_\mu^1 T_2^{40\mu}$, $\partial_\mu^1 \partial_\nu^2 T_2^{44\mu\nu}$ or $i\partial_\nu^2 T_2^{61\nu}$, due to $\square D^F(x_1 - x_2) = \delta(x_1 - x_2)$. It turns out that all these local terms are four-ghost interactions, which add up to

$$N_{uu\tilde{u}\tilde{u}}(x_1 - x_2) = -i g^2 \big(\tfrac{1}{2}(\beta_2)^2 + \beta_1 - 2\beta_1\beta_2\big) f_{abr} f_{cdr} \delta(x_1 - x_2) : u_a u_b \tilde{u}_c \tilde{u}_d : \tag{2.60}$$

in agreement with the much longer calculation in [13].

Remarks. (1) The powers of β_1, β_2 in (2.60) tell us the origin of the corresponding term. For example the term $\sim \beta_1\beta_2$ comes from $T_2^{27} + T_2^{72}$.

(2) We have seen that on the tree sector the normalizations of $T_2^{20}, \ldots, T_2^{72}$ are *uniquely* fixed by (2.16) or (2.34). However, this does not imply that gauge invariance fixes the normalization of $T_2^{20}|_{\text{tree}}, \ldots, T_2^{72}|_{\text{tree}}$ uniquely. The latter statement is a by-product of the calculation in [13].

(3) In agreement with our observations at first order (see remark (1) of section 1.2), *there is no ambiguity in the four-gluon interaction (2.59)—it is independent of* β_1, β_2.

(4) The most general coupling which is gauge invariant (2.14) up to all orders (this is not proved completely for the coboundary-coupling) has been given. It can be compared with the most general Lagrangian (written in terms of interacting fields) which is invariant under the full BRS transformations of the interacting fields—see equation (3.13) of [10]. For this purpose we must choose the Feynman gauge $\lambda = 1$ in this equation. Then one easily verifies that the terms $\sim g$ and $\sim g^2$ in the interaction part of this Lagrangian agree with $(T_1^0 + \beta_1 d_Q K_1 + \beta_2 \partial_\mu K_2^\mu) \sim g$ and with N_{AAAA}, $N_{uu\tilde{u}\tilde{u}} \sim g^2$, if we set $\beta_2 = 2\beta_1$ and identify the free parameter α of [10] with $\beta_2 = 2\beta_1$. There is only a one-parametric freedom in [10] which is given by adding to the Lagrangian $\alpha s(\cdots)$. The latter is a coboundary with respect to the BRS operator s. In doing so the Lagrangian remains s-invariant, due to the nilpotency of s. This seems to be analogous to our coboundary-coupling $\beta_1 d_Q K_1$ (1.14). However, we see from $\alpha = 2\beta_1 = \beta_2$ that there is not a complete correspondence: a change of α also means the addition of a divergence $\beta_2 \partial_\mu K_2^\mu$ (1.16). Since in our framework the interaction is switched off by $g \in \mathcal{S}(\mathbb{R}^4)$, our gauge invariance is not $[Q, T_n] = 0$ but $[Q, T_n] =$ (divergences), and, therefore, we have the freedom of adding a divergence-coupling (1.16) to T_1. This explains the fact that we have a two-parametric freedom and not only a one-parametric one.

(5) We call a normalization term $N_{\mathcal{O}}$ (2.58) 'natural', if there is a corresponding non-vanishing non-local term, more precisely if $T_{20}|_{\text{tree}}$ (2.57) contains a non-vanishing C-number distribution $t_{\mathcal{O}}$ (with the same $\mathcal{O}$). N_{AAAA} (2.59) is of this kind. It can be generated by

replacing

$$\partial^\mu\partial^\nu D^F(x_1-x_2) \qquad \text{by} \qquad \left[\partial^\mu\partial^\nu D^F(x_1-x_2)-\tfrac{1}{2}g^{\mu\nu}\delta(x_1-x_2)\right] \tag{2.61}$$

in t_{AAAA} [1, 5]. The other normalization terms are called 'unnatural', since they do not naturally arise in the inductive construction of the T_n's - the numerical distribution $d_{\mathcal{O}} = 0$ is split in $d_{\mathcal{O}}(x_1-x_2) = \delta^{(4)}(x_1-x_2) - \delta^{(4)}(x_1-x_2)$. $N_{uu\tilde{u}\tilde{u}}$ is unnatural, because in the corresponding diagram $\partial_\mu A_a^\mu(x_1)$ and $\partial_\nu A_b^\nu(x_2)$ are contracted, which gives $-i\delta_{ab}g^{\mu\nu}\partial_\mu\partial_\nu D_0^+(x_1-x_2)=0$. ($D_0^+(x_1-x_2)$ is the positive frequency part of the massless Pauli–Jordan distribution.) Note that the proof of gauge invariance (1.8) at higher orders $n \geqslant 3$ [2–5] uses normalizations which could be unnatural in an analogous sense.

2.8. Non-uniqueness of quantized Yang–Mills theories

To simplify the discussion we assume that (2.34) and (2.35) hold to any order. Then the ambiguities of quantized Yang–Mills theories, which are given by the free choice of the parameters $\beta_1, \beta_2 \in \mathbb{R}$, equations (1.14), (1.16), are not restricted by gauge invariance at higher orders, due to corollary 3 and (2.35). The freedom is reduced to a one-parametric set if we admit only natural normalization terms for second-order tree diagrams

$$N_{uu\tilde{u}\tilde{u}} = 0 \quad \Longleftrightarrow \quad \beta_1 = \frac{(\beta_2)^2}{4\beta_2-2} \qquad \beta_2 \neq \frac{1}{2}. \tag{2.62}$$

This prescription partially agrees with the Faddeev–Popov procedure: the exponentiation of a determinant can generate only terms quadratic in the ghosts. Therefore, the Faddeev–Popov method cannot yield a four-ghost interaction.

There is a more technical criterion which gives another restriction of the ambiguities and roughly speaking requires that the cancellations in the gauge invariance equation (2.14) be *simple*. To be more precise let us consider this equation for second-order tree diagrams. In the natural operator decomposition [5] the terms $\sim \partial^\mu\delta(x_1-x_2)$ cancel completely iff

$$\beta_2 = 0. \tag{2.63}$$

(For $\beta_2 \neq 0$ the terms $\sim \partial\delta : \mathcal{O} :$ must be combined with terms $\sim \delta : \mathcal{O}' :$, where the difference between the two operator combinations $\mathcal{O}'$ and $\mathcal{O}$ is that $\mathcal{O}'$ has one derivative more.) Let us assume that one can prove C-number identities (called 'Cg-identities' [2–5]) which express gauge invariance (2.14). Then the transition from the natural operator decomposition of (2.14) to the Cg-operator decomposition (i.e. the op. dec. in which the Cg-identities hold) is much more complicated for $\beta_2 \neq 0$ than for $\beta_1 = 0 = \beta_2$ [5]. We see from (2.62), (2.63) that *the theory with $\beta_1 = 0 = \beta_2$ is the simplest one*. However, this does not exclude the other values of β_1, β_2, since we can construct a Lorentz-, SU(N)- and P-, T-, C-invariant, (re)normalizable, gauge-invariant and pseudo-unitary S-matrix for any choice of $\beta_1, \beta_2 \in \mathbb{R}$.

We turn to the physical consequences of the freedom in the choice of β_1, β_2. For this purpose we consider $PT_n(x_1,\ldots,x_n)P$, where T_n is given by (2.10) and P is the projector on the physical subspace [4]. By means of $d_Q A_a^\mu = i\partial^\mu u_a$, $d_Q u_a = 0$, $d_Q\tilde{u}_a = -i\partial_\nu A_a^\nu$ and the fact that $\partial^\mu u_a$ and $\partial_\nu A_a^\nu$ are unphysical fields, we conclude that

$$P d_Q F_n(x_1,\ldots,x_n)P = 0 \tag{2.64}$$

where $F = A', R', R'', D, A, R, T', T, \tilde{T}$. Together with propositions 1 and 4 (equations (2.16), (2.34)), we obtain

$$PT_n(x_1,\ldots,x_n)P = T_n^{0\ldots 0}(x_1,\ldots,x_n) + (\text{sum of divergences}). \tag{2.65}$$

On the right-hand side *the dependence on β_1, β_2 is exclusively in the divergences*. However, the infrared behaviour of Yang–Mills theories is not under control. Therefore, we cannot conclude that the divergences in (2.65) vanish in the adiabatic limit $g \to 1$.

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Appendix A. Inductive construction of the T_n's according to Epstein and Glaser

The inputs to the inductive construction of the T_n's (1.1) are the T_1^i's (e.g. equations (1.2)–(1.4), (1.7), (2.2)–(2.7)) in terms of *free fields*. The couplings T_1^i are roughly speaking given by the interaction Lagrangian densities. Let us summarize the inductive step as a recipe. For the derivation of this construction from causality and translation invariance (only these two requirements are needed) we refer the reader to [6, 7]. In analogy with (1.1) we denote the n-point distributions of the inverse S-matrix $S(g_0, \ldots, g_l)^{-1}$ by $\tilde{T}_n(x_1, \ldots, x_n)$. Having constructed all $T_k, \tilde{T}_k$ for lower orders $k \leqslant n-1$, we can define the operator-valued distributions $R'_n,\ A'_n,\ R''_n$, which are sums of tensor products:

$$R'_n(x_1, \ldots; x_n) \stackrel{\text{def}}{=} \sum_{X,Y} T_{n-k}(Y, x_n)\tilde{T}_k(X) \tag{A.1}$$

$$A'_n(x_1, \ldots; x_n) \stackrel{\text{def}}{=} \sum_{X,Y} \tilde{T}_k(X)T_{n-k}(Y, x_n) \tag{A.2}$$

$$R''_n(x_1, \ldots; x_n) \stackrel{\text{def}}{=} \sum_{X,Y} T_k(X)\tilde{T}_{n-k}(Y, x_n) \tag{A.3}$$

where $X \stackrel{\text{def}}{=} \{x_{i_1}, \ldots, x_{i_k}\},\ Y \stackrel{\text{def}}{=} \{x_{i_{k+1}}, \ldots, x_{i_{n-1}}\},\ X \cup Y = \{x_1, \ldots, x_{n-1}\}$ and the sum is over all partitions of this kind with $1 \leqslant k \equiv | X | \leqslant n-1$. In order to simplify the notation, the Lorentz indices and the upper indices i_s denoting the kind of vertex $T_1^{i_s}(x_s)$ (see, e.g., equations (2.1)–(2.7)) are omitted. No confusion should arise, since i_s is strictly coupled to the spacetime argument x_s. One can prove that

$$D_n \stackrel{\text{def}}{=} R'_n - A'_n \tag{A.4}$$

has causal support

$$\operatorname{supp} D_n(x_1, \ldots; x_n) \subset (\Gamma^+_{n-1}(x_n) \cup \Gamma^-_{n-1}(x_n)) \tag{A.5}$$

where

$$\Gamma^\pm_{n-1}(x_n) \stackrel{\text{def}}{=} \{(x_1, \ldots, x_n) \in \mathbb{R}^{4n} | x_j \in x_n + \bar{V}^\pm,\ \forall\ j = 1, \ldots, n-1\}. \tag{A.6}$$

The crucial step in the inductive construction is the *correct distribution splitting* of D_n:

$$D_n = R_n - A_n \tag{A.7}$$

with

$$\operatorname{supp} R_n(x_1, \ldots; x_n) \subset \Gamma^+_{n-1}(x_n) \qquad \text{and} \qquad \operatorname{supp} A_n(x_1, \ldots; x_n) \subset \Gamma^-_{n-1}(x_n). \tag{A.8}$$

For this purpose we expand the operator-valued distributions in the normally ordered form:

$$F_n(x_1, \ldots, x_n) = \sum_{\mathcal{O}} f_{\mathcal{O}}(x_1 - x_n, \ldots, x_{n-1} - x_n) : \mathcal{O}(x_1, \ldots, x_n) : \tag{A.9}$$

where $F = R', A', D, R, A, T, \tilde{T}$ and $\mathcal{O}(x_1, \ldots, x_n)$ is a combination of the free field operators. The coefficients $f_{\mathcal{O}}$ are C-number distributions. Due to translation invariance, they depend only on the relative coordinates and, therefore, are responsible for the support properties. Consequently, the splitting must be done in these C-number distributions. Obviously, the critical point for the splitting is the UV-point

$$\Gamma^+_{n-1}(x_n) \cap \Gamma^-_{n-1}(x_n) = \{(x_1, \ldots, x_n) \in \mathbb{R}^{4n} | x_1 = x_2 = \cdots = x_n\}. \tag{A.10}$$

In order to measure the behaviour of the C-number distribution f in the vicinity of this point, one defines an index $\omega(f)$, which is called the *singular order of f at $x = 0$* [6, 7]. We will need the following example. Let D^a, $a \stackrel{\text{def}}{=} (a_1, \ldots, a_m)$, be a partial differential operator. Then

$$\omega(D^a \delta^{(m)}(x_1, \ldots, x_m)) = |a| \stackrel{\text{def}}{=} a_1 + \cdots + a_m. \tag{A.11}$$

If $\omega(d_{\mathcal{O}}) < 0$, the splitting of $d_{\mathcal{O}}$ is trivial and uniquely given by multiplication with a step function [6, 7].

If $\omega(d_{\mathcal{O}}) \geqslant 0$, one must perform the splitting more carefully [6, 7]. Moreover, it is not unique. One has an undetermined polynomial which is of degree $\omega(d_{\mathcal{O}})$ (the degree cannot be higher since renormalizability requires $\omega(r_{\mathcal{O}}) = \omega(d_{\mathcal{O}})$),

$$r_{\mathcal{O}}(x_1 - x_n, \ldots, x_{n-1} - x_n) = r^0_{\mathcal{O}}(\cdots) + \sum_{|a|=0}^{\omega(d_{\mathcal{O}})} C_a D^a \delta^{(4(n-1))}(x_1 - x_n, \ldots, x_{n-1} - x_n) \tag{A.12}$$

where $r^0_{\mathcal{O}}$ is a special splitting solution and C_a are the undetermined normalization constants. If one also performs the splitting in this case by multiplying with a step function, one obtains the usual, UV-divergent Feynman rules. However, this procedure is mathematically inconsistent. The correct distribution splitting saves us from UV-divergences.

From R_n one constructs

$$T'_n \stackrel{\text{def}}{=} R_n - R'_n \tag{A.13}$$

and T_n is obtained by symmetrization of T'_n

$$T_n^{i_1 \ldots i_n}(x_1, \ldots, x_n) = \sum_{\pi \in \mathcal{S}_n} \frac{1}{n!} T'^{\, i_{\pi 1} \ldots i_{\pi n}}_n(x_{\pi 1}, \ldots, x_{\pi n}). \tag{A.14}$$

In order to finish the inductive step we must construct

$$\tilde{T}_n \stackrel{\text{def}}{=} -T_n - R'_n - R''_n. \tag{A.15}$$

One can prove that (A.14), (A15) are the correct n-point distributions of $S(g_0, \ldots, g_l)$ (1.1) and $S(g_0, \ldots, g_l)^{-1}$, respectively, fulfilling the requirements of causality and translation invariance. Note that

$$\omega \stackrel{\text{def}}{=} \omega(t_{\mathcal{O}}) = \omega(r_{\mathcal{O}}) = \omega(d_{\mathcal{O}}). \tag{A.16}$$

The undetermined local terms (A.12) go over from $r_{\mathcal{O}}$ to $t_{\mathcal{O}}$. The normalization constants C_a are restricted by Lorentz- and SU(N)-invariance, the permutation symmetry (2.19), discrete symmetries, pseudo-unitarity and gauge invariance (cf section 1.2). The latter restriction plays an important role in this paper.

In our Yang–Mills model one can prove by means of scaling properties [7] that

$$\omega \leqslant \omega(\mathcal{O}) \stackrel{\text{def}}{=} 4 - b - g - d \tag{A.17}$$

where b is the number of gauge bosons (A, F), g the number of ghosts $(u, \tilde{u})$ and d the number of derivatives $(F, \partial\tilde{u}, \ldots)$ in $\mathcal{O}$. The proof of (A.17) in [2] is written for $T_n^{0\ldots 0}$ and $T_n^{10\ldots 0}$. However, it goes through without change for all $T_n^{i_1\ldots i_n}$ with $i_1,\ldots,i_n \in \{0,1,2,3,5,7\}$ (see equations (2.1)–(2.7) for the notation), especially for the physically relevant T_n (2.10). The couplings T_1^4 and T_1^6 have mass dimension 3 instead of 4. Therefore, there exists a lower upper bound $\tilde{\omega}(\mathcal{O})$ for the singular order ω of diagrams with at least one vertex T_1^4 or T_1^6: $\omega \leqslant \tilde{\omega}(\mathcal{O}) < \omega(\mathcal{O}) = 4 - b - g - d$. The fact that ω is bounded in the order n of the perturbation series (here it is even independent of n) is the *(re)normalizability* of the model.

Appendix B. Proof of equation (2.39)

Since equation (2.39) is a gauge invariance equation, it can be violated only in the splitting (A.7) and solely by local terms. No vacuum diagrams appear in (2.39).

B.1. Tree diagrams

We work with the technique of [1]. The splitting $D_2^{\cdots}|_{\mathrm{tree}} \to R_{20}^{\cdots}|_{\mathrm{tree}}$ is done by replacing $D_0(x_1-x_2)$ (which is the mass zero Pauli–Jordan distribution) with its retarded part $D_0^{ret}(x_1-x_2)$ everywhere. As in (2.57), the lower index 0 in $R_{20}^{\cdots}$ and in $T_{20}^{\cdots} \stackrel{\mathrm{def}}{=} R_{20}^{\cdots} - R_2'^{\cdots}$ (A.13), (A14) indicates this special normalization in the tree sector. Note $\Box D_0^{ret} = \delta^{(4)}$, in contrast to $\Box D_0 = 0$. This is the reason for the appearance of local terms A^ν which destroy (2.39)

$$d_Q R_{20}^{10\nu}|_{\mathrm{tree}} = i\partial_\mu^1 R_{20}^{50\nu\mu}|_{\mathrm{tree}} - i\partial_\mu^2 R_{20}^{11\nu\mu}|_{\mathrm{tree}} - A^\nu. \tag{B.1}$$

Picking out all local terms (they all are generated in the divergences on the right-hand side due to $\Box D_0^{ret} = \delta^{(4)}$) one finds that

$$\begin{aligned} A^\nu(x_1,x_2) = -g^2 f_{abr} f_{cdr} \Big\{ &\tfrac{1}{2}\delta(x_1-x_2) : u_a u_b A_{\mu c} F_d^{\mu\nu} : \\ &+ \tfrac{1}{2}\partial^\mu \delta(x_1-x_2) : u_a(x_1) u_b(x_1) A_{\mu c}(x_2) A_d^\nu(x_2) : \\ &+ [g^{\tau\mu}\partial^\nu \delta(x_1-x_2) - g^{\nu\mu}\partial^\tau \delta(x_1-x_2)] : A_{\tau a}(x_1) u_b(x_1) A_{\mu c}(x_2) u_d(x_2) : \Big\} \\ = i\partial_\mu^1 B^{\nu\mu}&(x_1,x_2) + i\partial_\mu^2 B^{\nu\mu}(x_1,x_2) + d_Q N^\nu(x_1,x_2) \end{aligned} \tag{B.2}$$

with

$$B^{\nu\mu}(x_1,x_2) \stackrel{\mathrm{def}}{=} \tfrac{1}{2} i g^2 f_{abr} f_{cdr} \delta(x_1-x_2) : u_a u_b A_c^\mu A_d^\nu : \tag{B.3}$$

$$N^\nu(x_1,x_2) \stackrel{\mathrm{def}}{=} -i g^2 f_{abr} f_{cdr} \delta(x_1-x_2) : A_{\mu a} u_b A_c^\mu A_d^\nu : . \tag{B.4}$$

(Note that a term $\sim f_{abr} f_{cdr} \delta(x_1-x_2) : u_a u_b u_c \partial^\nu \tilde{u}_d :$ vanishes due to the antisymmetry of the operator part in a, b, c and the Jacobi identity for the $f_{\ldots}$'s.) Obviously the symmetries $T_{20}^{50\nu\mu}(x_1,x_2) = -T_{20}^{50\mu\nu}(x_1,x_2)$ and $T_{20}^{11\nu\mu}(x_1,x_2) = -T_{20}^{11\mu\nu}(x_2,x_1)$ are preserved in the finite renormalizations

$$T_2^{50\nu\mu} \stackrel{\mathrm{def}}{=} T_{20}^{50\nu\mu} - B^{\nu\mu} \tag{B.5}$$

$$T_2^{11\nu\mu} \stackrel{\mathrm{def}}{=} T_{20}^{11\nu\mu} + B^{\nu\mu} \tag{B.6}$$

and

$$T_2^{10\nu} \stackrel{\mathrm{def}}{=} T_{20}^{10\nu} + N^\nu. \tag{B.7}$$

Due to (B.1), (B2), these $T_2^{\cdot\cdot}$-distributions (B.5)–(B7) satisfy (2.39) on tree level, and one easily verifies that (2.39) fixes the normalization of $T_2^{10}|_{\text{tree}}$ *uniquely*.

On the other hand the normalization in the tree sector of $T_2^{10\nu}(x_1, x_2) = T_2^{01\nu}(x_2, x_1)$ is uniquely determined by gauge invariance (1.8) at second order (see section 3.2 of [5])

$$d_Q T_2^{00} = i\partial_\nu^1 T_2^{10\nu} + i\partial_\nu^2 T_2^{01\nu} \tag{B.8}$$

where $T_2^{00}|_{\text{tree}}$ is normalized by (2.59) (four-gluon interaction). These two normalizations of $T_2^{10}|_{\text{tree}}$ (equations (B.7) and (B.8)) agree exactly.

B.2. Two-leg diagrams

We denote the numerical two-leg distributions in the following way:

$$F_2^{10\nu}(x_1, x_2)\big|_{\text{2-leg}} = f_{uA}^{10\nu\mu}(x_1 - x_2) : u_a(x_1)A_{\mu a}(x_2) :$$

$$+ f_{Au}^{10\nu\mu}(x_1 - x_2) : A_{\mu a}(x_1)u_a(x_2) : + \cdots : uF : + \cdots : Fu : \tag{B.9}$$

$$F_2^{50\nu\mu}(x_1, x_2)\big|_{\text{2-leg}} = f_{uu}^{50\nu\mu}(x_1 - x_2) : u_a(x_1)u_a(x_2) : \tag{B.10}$$

$$F_2^{11\nu\mu}(x_1, x_2)\big|_{\text{2-leg}} = f_{uu}^{11\nu\mu}(x_1 - x_2) : u_a(x_1)u_a(x_2) : \tag{B.11}$$

for $(F, f) = (T, t),\ (D, d), \ldots$. Again we choose a normalization of $T_2^{50\nu\mu}\big|_{\text{2-leg}}$ which is antisymmetrical in $\nu \leftrightarrow \mu$. Together with the fact that there exists no Lorentz covariant, antisymmetric tensor of second rank which depends on one Lorentz vector only, we conclude that

$$t_{uu}^{50\nu\mu} = 0. \tag{B.12}$$

Since $T_2^{10\nu}$ also appears in (B.8), we have some information about $t_{uA}^{10\nu\mu}$, $t_{Au}^{10\nu\mu}$ (B.9) from the C-number identities expressing (B.8) [2], namely

$$t_{Au}^{10\nu\mu} = -t_{Au}^{10\mu\nu} \qquad \text{and therefore} \qquad t_{Au}^{10\nu\mu} = 0 \tag{B.13}$$

$$\partial_\nu^1 t_{uA}^{10\nu\mu} = 0 \tag{B.14}$$

$$t_{uA}^{10\nu\mu} = t_{AA}^{00\nu\mu} \qquad \text{and therefore} \qquad t_{uA}^{10\nu\mu}(y) = t_{uA}^{10\mu\nu}(-y) \tag{B.15}$$

where $t_{AA}^{00\nu\mu}(x_1 - x_2)$ is the C-number distribution which belongs to the operators $: A_{\nu a}(x_1) A_{\mu a}(x_2) :$ in $T_2^{00}(x_1, x_2)$. Note that $d_{uu}^{11\nu\mu}$ has exactly the same (amputated) diagrams as $d_{uA}^{10\nu\mu}$, consequently $d_{uu}^{11\nu\mu} = d_{uA}^{10\nu\mu}$. If we split $d_{uu}^{11\nu\mu}$ in the same way as $d_{uA}^{10\nu\mu}$, we obtain

$$t_{uu}^{11\nu\mu} = t_{uA}^{10\nu\mu}. \tag{B.16}$$

Obviously equations (B.12)–(B16) also hold for t replaced by $\tilde{t}$. Inserting equations (B.9)–(B16) into (2.39) we see that (2.39) is also fulfilled on the two-leg sector. □

Appendix C. Coboundary-coupling at arbitrary order

To shorten the notation we shall omit the Lorentz indices and define

$$\mathcal{S}_r F_n^{67\ldots7i_{r+1}\ldots i_n} \stackrel{\text{def}}{=} \frac{1}{r}[F_n^{67\ldots7i_{r+1}\ldots i_n} + F_n^{767\ldots7i_{r+1}\ldots i_n} + \cdots + F_n^{7\ldots76i_{r+1}\ldots i_n}] \tag{C.1}$$

where $F = T,\ \tilde{T}$.

Proposition 5. Assuming that the identities (2.50*a*) hold, the following equations are simultaneously fulfilled to all orders $n \in \mathbb{N}$ for $F = T, \tilde{T}$, if suitable symmetrical normalizations are chosen:

$$d_Q F_n^{7...75...51...10...0} = i \sum_{j=r+t+1}^{r+t+s} (-1)^{(j-r-t-1)} \partial^j F_n^{7...75...51...151...10...0}$$

$$+ i(-1)^s \sum_{j=r+t+s+1}^{n} \partial^j F_n^{7...75...51...10...010...0}$$

$$0 \leqslant r, t, s \leqslant n \quad r+t+s \leqslant n \tag{C.2}$$

and

$$d_Q \mathcal{S}_r F_n^{67...75...51...10...0} = i \sum_{j=r+t+1}^{r+t+s} (-1)^{(j-r-t)} \partial^j \mathcal{S}_r F_n^{67...75...51...151...10...0}$$

$$+ i(-1)^{s+1} \sum_{j=r+t+s+1}^{n} \partial^j \mathcal{S}_r F_n^{67...75...51...10...010...0} + F_n^{7...75...51...10...0}$$

$$1 \leqslant r \leqslant n \quad 0 \leqslant t, s \leqslant n \quad r+t+s \leqslant n \tag{C.3}$$

where the $F_n^{7...75...51...10...0}$ ($F_n^{67...75...51...10...0}$) on the left-hand sides have t indices 5, s indices 1 and r indices 7 (($r-1$) indices 7 and one index 6). All derivatives on the right-hand sides are divergences.

Note that equation (C.2) is a generalization of gauge invariance (2.35) and (1.8); the representations (2.34) and (2.49) are special cases of (C.3). The indices may be permuted in (C.2), (C3) according to (2.19).

Proof. The reasoning runs essentially along the same lines as that of proposition 4. Therefore, we only give an outline of it. First we consider (C.3). We start with (A.2):

$$d_Q A_n'^{67...75...51...10...0} = \sum [(d_Q \tilde{T}_k^{\cdots}) T_{n-k}^{\cdots} \pm \tilde{T}_k^{\cdots} d_Q T_{n-k}^{\cdots}]. \tag{C.4}$$

The upper indices of $\tilde{T}_k$ and T_{n-k} on the right-hand side are an arbitrary number of indices 7,5,1,0 and at most one index 6. Consequently, we can insert the induction hypothesis (C.2), (C3) for $d_Q \tilde{T}_k$ and $d_Q T_{n-k}$ and obtain (C.3) for the A_n'-distributions, and similarly for R_n', R_n''. Therefore, we may define the normalization of $R_n^{7...75...51...10...0}$ by (C.3). This procedure conserves (C.3) in the splitting (A.7), and the remaining steps do not destroy it either.

We turn to (C.2). The case $r = 0$ is the assumption (2.50*a*). For $1 \leqslant r \leqslant n$ we apply d_Q to (C.3) and use $(d_Q)^2 = 0$:

$$d_Q F_n^{7...75...51...10...0} = -i \sum_{j=r+t+1}^{r+t+s} (-1)^{(j-r-t)} \partial^j d_Q \mathcal{S}_r F_n^{67...75...51...151...10...0} \tag{C5a}$$

$$- i(-1)^{s+1} \sum_{j=r+t+s+1}^{n} \partial^j d_Q \mathcal{S}_r F_n^{67...75...51...10...010...0}. \tag{C5b}$$

Next we again insert (C.3) into both terms on the right-hand side:

$$(\text{C.5}a) = -i \sum_{j=r+t+1}^{r+t+s} (-1)^{(j-r-t)} \partial^j \Bigg\{ i \sum_{l=r+t+1\ (l\neq j)}^{r+t+s} \pm \partial^l \mathcal{S}_r F_n^{67\ldots 75\ldots 51\ldots 151\ldots 151\ldots 10\ldots 0} \qquad (\text{C6}a)$$

$$+ i(-1)^s \sum_{l=r+t+s+1}^{n} \partial^l \mathcal{S}_r F_n^{67\ldots 75\ldots 51\ldots 151\ldots 10\ldots 010\ldots 0} \qquad (\text{C6}b)$$

$$+ F_n^{7\ldots 75\ldots 51\ldots 151\ldots 10\ldots 0} \Bigg\} \qquad (\text{C6}c)$$

$$(\text{C.5}b) = -i(-1)^{s+1} \sum_{j=r+t+s+1}^{n} \partial^j \Bigg\{ i \sum_{l=r+t+1}^{r+t+s} (-1)^{(l-r-t)} \partial^l \mathcal{S}_r F_n^{67\ldots 75\ldots 51\ldots 151\ldots 10\ldots 010\ldots 0} \qquad (\text{C7}a)$$

$$+ i(-1)^{s+1} \partial^j \mathcal{S}_r F_n^{67\ldots 75\ldots 51\ldots 10\ldots 050\ldots 0} \qquad (\text{C7}b)$$

$$+ i(-1)^{s+1} \sum_{l=r+t+s+1\ (l\neq j)}^{n} \pm \partial^l \mathcal{S}_r F_n^{67\ldots 75\ldots 51\ldots 10\ldots 010\ldots 010\ldots 0} \qquad (\text{C7}c)$$

$$+ F_n^{7\ldots 75\ldots 51\ldots 10\ldots 010\ldots 0} \Bigg\}. \qquad (\text{C7}d)$$

Equations (C.6b) and (C.7a) cancel. In a manner similar to the reasoning after (2.50), the terms (C.7b) and (C.7c) vanish because of $F_n^{\ldots 5\ldots \nu\mu} = -F_n^{\ldots 5\ldots \mu\nu}$ and the different signs of the (j,l)- and the (l,j)-term in $\sum_{j,l\ (j\neq l)} \pm \partial^j \partial^l F_n^{\ldots 010\ldots 010\ldots}$. The latter argument also applies to (C.6a). (Due to (1.11) the $\pm$ in (C.6a) is a factor $(-1)^{(l-r-t)}$ if $l < j$, and a sign $(-1)^{(l-r-t-1)}$ for $l > j$.) The expression $d_Q F_n^{7\ldots 75\ldots 51\ldots 10\ldots 0} = (\text{C.6}c) + (\text{C.7}d)$ remains, which is the assertion (C.2). □

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